# Some Remarks on the Oscillator Group

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Differential Geometry and its Applications
Brno, Czech Republic, August 19-23, 2013

# Outline

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# The oscillator group Osc

## Matrix representation

$$\text{Osc}: \quad \begin{bmatrix} 1 & -y\cos\theta - z\sin\theta & z\cos\theta - y\sin\theta & -2x \\ 0 & \cos\theta & \sin\theta & z \\ 0 & -\sin\theta & \cos\theta & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(x, y, z, \theta \in \mathbb{R})$$

## Fact (Medina, 1985)

Osc is the only simply connected four-dimensional non-Abelian solvable (matrix) Lie group which admits a bi-invariant Lorentzian metric.

# The oscillator Lie algebra $\mathfrak{osc}$

## Matrix representation

$$\mathfrak{osc}: \quad \begin{bmatrix} 0 & -y & z & -2x \\ 0 & 0 & \theta & z \\ 0 & -\theta & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (x, y, z, \theta \in \mathbb{R})$$

## Commutator table for standard basis

| | $E_1$ | $E_2$ | $E_3$ | $E_4$ |
|---|---|---|---|---|
| $E_1$ | 0 | 0 | 0 | 0 |
| $E_2$ | 0 | 0 | $E_1$ | $-E_3$ |
| $E_3$ | 0 | $-E_1$ | 0 | $E_2$ |
| $E_4$ | 0 | $E_3$ | $-E_2$ | 0 |

# Adjoint orbits

## Proposition

*The adjoint orbit* $\mathcal{O}_A = \{\mathrm{Ad}_g A \,:\, g \in \mathsf{Osc}\}$ *through* $A = M(x^0, y^0, z^0, \theta^0)$ *is (in the hyperplane* $\theta^0 E_4 + \langle E_1, E_2, E_3 \rangle$*)*

- *a point* $\{x^0 E_1\}$ $\qquad (y^0 = z^0 = \theta^0 = 0)$
- *a cylinder*
  $\{x E_1 + r\cos\vartheta E_2 + r\sin\vartheta E_3 \,:\, x, \vartheta \in \mathbb{R},\ r = \sqrt{(y^0)^2 + (z^0)^2}\}$
  $(\theta^0 = 0,\ (y^0)^2 + (z^0)^2 \neq 0)$
- *a paraboloid* $\{* E_1 + r\cos\vartheta E_2 + r\sin\vartheta E_3 + \theta^0 E_4 \,:\, r, \vartheta \in \mathbb{R}\}$,
  $* = \frac{1}{2\theta}(2x^0\theta^0 + (y^0)^2 + (z^0)^2 - r^2) \qquad (\theta^0 \neq 0).$

# Invariant scalar products

A scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ on (a Lie algebra) $\mathfrak{g}$ is said to be invariant if

$$\langle\langle [A, B], C\rangle\rangle = \langle\langle A, [B, C]\rangle\rangle, \qquad A, B, C \in \mathfrak{g}$$

or, equivalently, $\langle\langle \mathrm{Ad}_g A \,, \mathrm{Ad}_g\, B\rangle\rangle = \langle\langle A, B\rangle\rangle, \qquad A, B \in \mathfrak{g},\ g \in \mathsf{G}.$

## Proposition

*The Lie algebra $\mathfrak{osc}$ admits (exactly) one family of invariant scalar products $\omega_\alpha$, $\alpha \in \mathbb{R}$; in coordinates,*

$$\omega_\alpha = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \alpha \end{bmatrix}.$$

# Coadjoint orbits (and Casimirs)

## The dual space $\mathfrak{osc}^*$

The dual space $\mathfrak{osc}^*$ can be (canonically) identified with $\mathfrak{osc}$ via $\omega_\alpha$:

$$\omega_\alpha(A, \cdot) \in \mathfrak{osc}^* \quad \longleftrightarrow \quad A \in \mathfrak{osc}.$$

## Note

- The coadjoint orbits of $\mathfrak{osc}$ are the images of the adjoint orbits, under (any of) the isomorphisms $A \mapsto \omega_\alpha(A, \cdot)$.
- $\omega_\alpha$ induces a family of Casimir functions on the (minus) Lie-Poisson space $\mathfrak{osc}^*_-$:

$$C_\alpha : \mathfrak{osc}^* \to \mathbb{R}, \quad \omega_\alpha(A, \cdot) \mapsto \omega_\alpha(A, A)$$

$$C_\alpha(p) = -\alpha\, p_1^2 + p_2^2 + p_3^2 + 2p_1 p_4 \qquad (p = p_i E^i).$$

# Equivalence (of linear subspaces)

## Definition

Two linear subspaces $\mathfrak{a}$ and $\mathfrak{b}$ of (a Lie algebra) $\mathfrak{g}$ are $\mathfrak{L}$-equivalent if there exists an automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that

$$\psi \cdot \mathfrak{a} = \mathfrak{b}.$$

## Note

The trace $\Gamma = \langle B_1, \ldots, B_\ell \rangle$ of a (left-invariant) control affine system

$$(\Sigma) \qquad \dot{g} = g(u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

on (the matrix Lie group) $\mathsf{G}$ is a linear subspace of (the Lie algebra) $\mathfrak{g}$.
*Two (full-rank) control affine systems are (detached feedback) equivalent if and only if their corresponding traces are $\mathfrak{L}$-equivalent.*

# Automorphisms

## The group of automorphisms of $\mathfrak{osc}$: $\mathrm{Aut}\,(\mathfrak{osc})$

$$\left\{ \begin{bmatrix} \sigma(x^2+y^2) & wy-\sigma vx & -wx-\sigma vy & u \\ 0 & x & y & v \\ 0 & -\sigma y & \sigma x & w \\ 0 & 0 & 0 & \sigma \end{bmatrix} : x, y, u, v, w, \sigma \in \mathbb{R} \right\}$$

$$(x^2+y^2 \neq 0,\ |\sigma| = 1).$$

## Note

For inner automorphisms $\mathrm{Inn}\,(\mathfrak{osc})$, we require

$$x^2+y^2 = 1 \qquad \sigma = 1 \qquad u = -\tfrac{1}{2}(v^2+w^2).$$

# Classification of linear subspaces

## Theorem

*Any proper linear subspace of (the Lie algebra) $\mathfrak{osc}$ is $\mathfrak{L}$-equivalent to exactly one of the linear subspaces*

$$\langle E_1 \rangle, \qquad \langle E_2 \rangle, \qquad \langle E_4 \rangle$$
$$\langle E_1, E_2 \rangle, \qquad \langle E_1, E_4 \rangle, \qquad \langle E_2, E_3 \rangle, \qquad \langle E_2, E_4 \rangle$$
$$\langle E_1, E_2, E_3 \rangle, \qquad \langle E_1, E_2, E_4 \rangle, \qquad \langle E_2, E_3, E_4 \rangle.$$

# Classification of linear subspaces (cont.)

## Proof (sketch): the one-dimensional case

Let $\mathfrak{a} = \langle a^i E_i \rangle \subset \mathfrak{osc}$.

- If $a^4 \neq 0$, then we may assume $a^4 = 1$ and so

$$\psi = \begin{bmatrix} 1 & a^2 & a^3 & -a^1 - (a^2)^2 - (a^3)^2 \\ 0 & 1 & 0 & -a^2 \\ 0 & 0 & 1 & -a^3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}\,(\mathfrak{osc}), \quad \psi \cdot \mathfrak{a} = \langle E_4 \rangle.$$

- If $a^4 = 0$ and $\alpha = (a^2)^2 + (a^3)^2 \neq 0$, then

$$\psi = \frac{1}{\alpha} \begin{bmatrix} 1 & -\frac{a^1 a^2}{\alpha} & -\frac{a^1 a^3}{\alpha} & 0 \\ 0 & a^2 & a^3 & a^1 \\ 0 & -a^3 & a^2 & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix} \in \mathrm{Aut}\,(\mathfrak{osc}), \quad \psi \cdot \mathfrak{a} = \langle E_2 \rangle.$$

# Classification of linear subspaces (cont.)

## Proof (sketch): the three-dimensional case

Let $\mathfrak{a}^\perp = \langle a^i E_i \rangle \subset \mathfrak{osc}$. Suppose $a^4 \neq 0$.

- We have

$$\varphi = \begin{bmatrix} 1 & \frac{a^2}{a^4} & \frac{a^3}{a^4} & -\frac{(a^2)^2+(a^3)^2}{2(a^4)^2} \\ 0 & 1 & 0 & -\frac{a^2}{a^4} \\ 0 & 0 & 1 & -\frac{a^3}{a^4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Inn}\,(\mathfrak{osc}), \qquad \varphi \cdot \mathfrak{a}^\perp = \langle \alpha E_1 + E_4 \rangle.$$

- Thus $\varphi \cdot \mathfrak{a} = \langle E_2, E_3, -\alpha E_1 + E_4 \rangle$ and so

$$\psi = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}\,(\mathfrak{osc}), \qquad (\psi \circ \varphi) \cdot \mathfrak{a} = \langle E_2, E_3, E_4 \rangle.$$

# Classification of subalgebras

## Corollary

*Any proper subalgebra of (the Lie algebra) $\mathfrak{osc}$ is $\mathfrak{L}$-equivalent to exactly one of the subalgebras*

$$\langle E_1 \rangle, \qquad \langle E_2 \rangle, \qquad \langle E_4 \rangle$$
$$\langle E_1, E_2 \rangle, \qquad \langle E_1, E_4 \rangle$$
$$\langle E_1, E_2, E_3 \rangle.$$

## Note

- Except for $\langle E_1, E_2, E_3 \rangle$, all these subalgebras are Abelian.
- The Lie subalgebra $\langle E_1, E_2, E_3 \rangle$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_3$.

# Full-rank linear subspaces and ideals

**Corollary (full-rank linear subspaces)**

Any full-rank linear subspace of (the Lie algebra) $\mathfrak{osc}$ is $\mathfrak{L}$-equivalent to exactly one of the linear subspaces

$$\langle E_2, E_4\rangle, \qquad \langle E_1, E_2, E_4\rangle, \qquad \langle E_2, E_3, E_4\rangle.$$

**Corollary (ideals)**

The Lie algebra $\mathfrak{osc}$ has exactly two proper ideals:

$$\langle E_1\rangle = Z(\mathfrak{osc}), \qquad \langle E_1, E_2, E_3\rangle \cong \mathfrak{h}_3.$$

# Remark 1

## Remark (semidirect sum)

The Lie algebra $\mathfrak{osc}$ decomposes as semidirect sum of the ideal $\langle E_1, E_2, E_3 \rangle$ and the subalgebra $\langle E_4 \rangle$:

$$\mathfrak{osc} \cong \mathfrak{h}_3 \rtimes \mathbb{R}.$$

## Note

The Lie group $\mathrm{Osc}$ decomposes as semidirect product of (closed) Lie subgroups $\mathrm{H}_3$ and $\mathrm{SO}(2)$:

$$\mathrm{Osc} \cong \mathrm{H}_3 \rtimes \mathrm{SO}(2).$$

# Remark 2

## Remark (central extension)

The quotient of (the Lie algebra) $\mathfrak{osc}$ by the ideal $\langle E_1 \rangle$ is (isomorphic to) the Euclidean Lie algebra $\mathfrak{se}(2)$:

$$\mathfrak{osc}/Z(\mathfrak{osc}) \cong \mathfrak{se}(2).$$

## Note

- $\mathfrak{osc}$ is the only (non-trivial) four-dimensional central extension of $\mathfrak{se}(2)$.
- The Euclidean group $\mathsf{SE}(2)$ is (isomorphic to) a quotient group of (the Lie group) $\mathsf{Osc}$:

$$\mathsf{Osc}/Z(\mathsf{Osc}) \cong \mathsf{SE}(2).$$

# The universal covering Lie group

## Matrix representation

$$\widetilde{\mathrm{Osc}} : \quad \begin{bmatrix} 1 & -y\cos\theta - z\sin\theta & z\cos\theta - y\sin\theta & -2x & 0 \\ 0 & \cos\theta & \sin\theta & z & 0 \\ 0 & -\sin\theta & \cos\theta & y & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\theta} \end{bmatrix}$$

$$(x, y, z, \theta \in \mathbb{R})$$

# Normal discrete subgroups

## Center

$$Z(\widetilde{\mathrm{Osc}}) : \quad \begin{bmatrix} 1 & 0 & 0 & 0 & -2x \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{2\pi\theta} \end{bmatrix} \quad \longleftrightarrow \quad (x, \theta) \in \mathbb{R} \times \mathbb{Z} \subset \mathbb{R}^2$$

## Automorphisms of center

$$\mathrm{Aut}\,(\widetilde{\mathrm{Osc}})\Big|_{Z(\widetilde{\mathrm{Osc}})} = \left\{ \begin{bmatrix} x \\ \theta \end{bmatrix} \mapsto \begin{bmatrix} \sigma r & u \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} \ : \ r > 0, u \in \mathbb{R}, \sigma = \pm 1 \right\}$$

# Normal discrete subgroups (cont.)

**Lemma**

*Suppose* $x_1, x_2 \in \mathbb{R}$, $\theta_1, \theta_2 \in \mathbb{Z}$ *and* $(x_1, \theta_1)$, $(x_2, \theta_2)$ *are linearly independent. Then there exist* $r > 0$ *and* $u \in \mathbb{R}$ *such that*

$$\begin{bmatrix} r & u \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} \mathbb{Z} \right) = \begin{bmatrix} 0 \\ \gcd(\theta_1, \theta_2) \end{bmatrix} \mathbb{Z} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbb{Z}.$$

# Normal discrete subgroups (cont.)

## Proof (sketch)

- By Bézout's identity, there exists $a, b \in \mathbb{Z}$ such that

$$a\theta_1 + b\theta_2 = \gcd(\theta_1, \theta_2) > 0.$$

- For $r = \frac{a\theta_1 + b\theta_2}{|x_1\theta_2 - x_2\theta_1|}$ and $u = -\frac{ax_1 + bx_2}{|x_1\theta_2 - x_2\theta_1|}$ we have

$$\begin{bmatrix} r & u \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} \mathbb{Z} \right) = \begin{bmatrix} b\sigma \\ \theta_1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} -a\sigma \\ \theta_2 \end{bmatrix} \mathbb{Z}$$

  where $\sigma = \operatorname{sgn}(x_1\theta_2 - x_2\theta_1)$.
- By showing each group contains the generators of the other, we get

$$(b\sigma, \theta_1)\mathbb{Z} + (-a\sigma, \theta_2)\mathbb{Z} = (0, \gcd(\theta_1, \theta_2))\mathbb{Z} + (1, 0)\mathbb{Z}.$$

# Normal discrete subgroups: main result

**Theorem**

*Any (non-trivial) normal discrete subgroup of $\widetilde{Osc}$ is equivalent to exactly one of the following (central discrete subgroups):*

$$\mathsf{N}_{1,n} = \{\,(0, n\theta)\ :\ \theta \in \mathbb{Z}\}, \quad n \in \mathbb{N}$$

$$\mathsf{N}_2 = \{\,(x, 0)\ :\ x \in \mathbb{Z}\}$$

$$\mathsf{N}_{1,n} \times \mathsf{N}_2 = \{\,(x, n\theta)\ :\ \theta, x \in \mathbb{Z}\}, \quad n \in \mathbb{N}.$$

# Normal discrete subgroups: proof

## Proof (sketch)

Let N be a central discrete subgroup.

- If $\mathsf{N} = (0, n)\mathbb{Z}$, then $\mathsf{N} = \mathsf{N}_{1,n}$.
- If $\mathsf{N} = (r, 0)\mathbb{Z}$, then $\psi = \operatorname{diag}(\frac{1}{r}, 1) \in \mathsf{Aut}(\widetilde{\mathsf{Osc}})\big|_{Z(\widetilde{\mathsf{Osc}})}$, $\psi \cdot \mathsf{N} = \mathsf{N}_2$.
- If $\mathsf{N} = (r, n)\mathbb{Z}$, then

$$\psi = \begin{bmatrix} 1 & -\frac{r}{n} \\ 0 & 1 \end{bmatrix} \in \mathsf{Aut}(\widetilde{\mathsf{Osc}})\Big|_{Z(\widetilde{\mathsf{Osc}})}, \qquad \psi \cdot \mathsf{N} = (0, n)\mathbb{Z} = \mathsf{N}_{1,n}.$$

- If $\mathsf{N} = (x_1, \theta_1)\mathbb{Z} + (x_2, \theta_2)\mathbb{Z}$, then $\exists \psi \in \mathsf{Aut}(\widetilde{\mathsf{Osc}})\big|_{Z(\widetilde{\mathsf{Osc}})}$ such that $\psi \cdot \mathsf{N} = \mathsf{N}_{1,n} \times \mathsf{N}_2$, where $n = \gcd(\theta_1, \theta_2)$.

# Classification of connected Lie groups

## Corollary

*There are four types of connected Lie groups with associated Lie algebra (isomorphic to)* $\mathfrak{osc}$*:*

$$\widetilde{\mathrm{Osc}}, \qquad \widetilde{\mathrm{Osc}}/\mathrm{N}_{1,n}, \qquad \widetilde{\mathrm{Osc}}/\mathrm{N}_2, \qquad \widetilde{\mathrm{Osc}}/(\mathrm{N}_{1,n} \times \mathrm{N}_2).$$

## Note (covering morphisms)

- $\widetilde{\mathrm{Osc}}/\mathrm{N}_{1,n}$ is the $n$-fold covering of the oscillator group $\mathrm{Osc} \cong \widetilde{\mathrm{Osc}}/\mathrm{N}_{1,1}$.
- $\widetilde{\mathrm{Osc}}/\mathrm{N}_{1,n} \times \mathrm{N}_2$ is the $n$-fold covering of the group $\widetilde{\mathrm{Osc}}/(\mathrm{N}_{1,1} \times \mathrm{N}_2)$.

# Matrix Lie groups

## Fact (Onishchik–Vinberg, 1994)

A connected solvable Lie group is a matrix Lie group if and only if its commutator subgroup is simply connected.

## Proposition

- *The groups*

$$\widetilde{\mathrm{Osc}} \quad \textit{and} \quad \widetilde{\mathrm{Osc}}/\mathrm{N}_{1,n}$$

  *are matrix Lie groups.*
- *The groups*

$$\widetilde{\mathrm{Osc}}/\mathrm{N}_2 \quad \textit{and} \quad \widetilde{\mathrm{Osc}}/(\mathrm{N}_{1,n} \times \mathrm{N}_2)$$

  *are not matrix Lie groups.*

# Outlook

## Oscillator groups : promising geometric objects

- invariant Lorentzian metrics
- locally symmetric structures
- Lie bialgebra structures (on the oscillator Lie algebras)
- Einstein-Yang-Mills equations

## The oscillator group Osc

- invariant sub-Riemannian structures
- invariant control systems